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Ashok S Guha, Brishti Guha

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FUTURE TARGETS AND MULTIPLE EQUILIBRIA

Ashok S. Guha* **Brishti Guha†**
Jawaharlal Nehru University **Singapore Management University**

Abstract

Multiple Pareto-rankable equilibria may obtain in an overlapping generations model where consumers save to reach a fixed target. Existence and uniqueness conditions are discussed. The model displays excess consumption sensitivity to current income and perfect old-age insurance.

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1. Introduction

Consumers often aim at target consumption levels of specific goods. Consumer durables (houses, cars) frequently come in indivisible units and the consumer is often satiated with a single unit : he desires just one piece, no more and no less. Where such a good bulks large in the consumer's budget, changes in its price may have unusual repercussions on demand for other goods. The gross substitutability property may not obtain: a rise in the price of the target good may reduce the budget available for other goods and the consumer's demand for them. This affects the existence and uniqueness of market equilibrium.

An important example occurs when the consumer aims at a future consumption target – say a house to retire to or a fixed payment to an old age home that will support him. This affects his saving behavior, particularly his reactions to interest rate changes. If such behavior occurs on a large scale, equilibrium in the capital market, the interest rate and the level of output are affected in curious ways.

We focus on the possibility of multiple equilibria in such environments. We suggest that an economy may persist in a high-saving equilibrium even at very low interest rates, offering one possible explanation for syndromes such as Japan's over the last fifteen years. We discuss conditions for uniqueness and demonstrate a multiplicity of Pareto-rankable equilibria for some parameters. An implication is excess response of consumption to current income. While a violation of the permanent income hypothesis, this is in line with the fact that there *is* an "excess sensitivity" puzzle, particularly for developing countries where consumers often violate the permanent income hypothesis (see Rao [2005], Lavi [2003], Chakrabarty and Schmalenbach [2002]).

* School of International Studies, Jawaharlal Nehru University, New Delhi 110067, India. E-mail: ashoksanjayguha@hotmail.com.

†Corresponding author. Department of Economics, School of Economics and Social Sciences, Singapore Management University, 90 Stamford Road, Singapore 178903. E-mail : bguha@smu.edu.sg The authors acknowledge input from Kaushik Basu.

2.The Model

Assume a world of overlapping generations in which all individuals are identical. Population is constant with the number in each generation being normalized to one. Each individual lives for two periods. At the outset of the first, he borrows capital from the previous generation, now retired, and promises to repay principal and prenegotiated interest after the production process. He then produces according to a production function $y_t = f(k_t) + \varepsilon_t$, where ε_t is a production shock with zero mean. Thereafter, he makes the contractual repayment and allocates the remaining output between consumption c_t and saving in the light of his interest rate expectations and preferences. At the beginning of the next period, he lends the savings to the new generation of producers at the interest rate expected in that period and lives in retirement thereafter on his interest and capital. Agents are all risk-neutral.

As in all models in which population is constant and individuals exhaust their income over their life-cycle, there is no aggregate saving: however, the young save to finance their retirement.

Competition for capital between producers drives the contractual interest rate to the expected marginal product of this capital $f^{'}(k_t)$. This represents the non-random component of marginal product of capital, which can be fully forecast given the capital stock at the time of contracting (which is just the savings carried over from the previous period by the then working generation). Given rational expectations, savers form correct expectations of this contractual interest rate and base their savings decisions on it. Thus the contract is signed before the production shock hits. As savers are guaranteed the interest rate $f^{'}(k_t)$, they are fully insured against the production shock – whether positive or negative - which is borne accordingly by the current producing generation. Since all agents are risk-neutral, the current generation is willing nonetheless to guarantee the older generation an interest rate of $f^{'}(k_t)$ - the expected value of the production shock being zero.

Suppose now that the second period consumption of each individual is fixed at $\underline{a}$. Let the younger generation's consumption in period t – the first period of its life – be denoted by $c_{1,t}$ while the older's consumption is $c_{2,t}$ (the older generation's consumption in its youth was $c_{2,t-1}$). Because individuals are saving to meet fixed targets, we always have

$$c_{2,t} = \underline{a} = (1+f^{'}(k_t))s_{2,t-1} \tag{1}$$

After paying the older generation, the younger one also wants to save just enough to ensure a second period consumption of $\underline{a}$. Accordingly the production shock is entirely absorbed in $c_{1,t}$, the current consumption of the working generation. This excess sensitivity of current consumption to transitory income goes against the permanent income hypothesis, but is a characteristic of target-savers. The following equation describes the younger generation's behavior in period t+1, when *it* gets old:

$$c_{2,t+1} = \underline{a} = (1+f^{'}(k_{t+1})s_{2,t} \tag{2}$$

Moreover, we have

$$k_{t+1} = s_{2,t} \qquad \forall t \tag{3}$$

From (1), (2) and (3) we can easily check that

$$s_{2,t} = s_{2,t-1} = s\ ,\ \ k_t = k_{t+1} = k\ \ k_t = k_{t+1} = k\ ,\quad \text{s = k} \tag{4}$$

Thus the game is stationary, despite the uncertain production shock. If in any period, the shock is so adverse as to reduce the total income of the younger generation – after interest payments on the older generation's loan – below the minimum required to reach the second period target, the young will not save, but will instead consume everything left over after making interest payments. No capital will be available next period to sustain production and the economy will disappear. An economy can be wiped out by a sufficiently unfavorable shock if

$$f(k) - \varepsilon - kf^{'}(k) < \frac{a}{1+f^{'}(k)}$$

for any ε. As long as production shocks are bounded, there is always a parameter space where this will never happen. For simplicity, we assume that production shocks are uniformly distributed in the interval [-ε*, ε*], and for the rest of this paper concentrate on the case where even the worst shocks leave the young with enough to reach their second period target – so that the economy is in no danger of disappearing.

3. Equilibrium

The *supply* of savings from the old is given by

$$s = k = \frac{a}{1+r^*} \tag{5}$$

where r* is the pre-negotiated interest rate that savers will face in the beginning of the next period. The *demand* of the young for savings, or equivalently for capital, is given by

$$f^{'}(k) = r^* \tag{6}$$

Thus equilibrium capital stock solves the equation

$$k(1+f^{'}(k)) = \underline{a} \tag{7}$$

The supply curve is negatively interest-elastic and representable as a rectangular hyperbola in (s,r) space. It is asymptotic to the vertical axis and to the horizontal through r = -1; it intersects the horizontal axis at s = <u>a</u> .. The shape of the demand curve depends on the production function, and determines the number of intersections. Given diminishing marginal productivity of capital, the demand curve is also down-sloping so that multiple intersections are possible. We now prove some properties of the supply and demand curves and give examples of specific production functions which give rise to (a) a unique equilibrium, and (b) multiple equilibria.

Proposition 1: With neo-classical production functions, the demand curve always lies below the supply curve for sufficiently small s: and if there is no capital saturation (f'(k) > 0 for all k), there is at least one intersection.

Proof : If, as k → 0, the limiting elasticity of capital-labour substitution is less than one, r converges to a finite ceiling[‡], so that the demand curve intercepts the vertical axis at a finite r. In this case clearly the demand curve lies below the supply curve (which is asymptotic to the vertical axis) for sufficiently small k (or s). If the elasticity of

[‡] For proof see Guha (1963).

substitution in the limit exceeds one, as k → 0, the relative share of capital in total output goes to zero while output tends to a finite limit[§]; if the elasticity equals one, the relative capital share remains constant while output goes to zero. In either event, the absolute share of capital sinks to zero This implies that the area below the demand curve would go to zero, while that below the supply curve goes to a positive limit, $\underline{a}$, as $r \to \infty$. Thus, the demand curve must lie below the supply curve as $r \to \infty$ (or as k → 0) whatever the elasticity of substitution.. Now suppose there is no capital saturation, so that the demand curve never intercepts the horizontal axis. In this case, clearly the demand curve lies above the supply curve for large enough values of k, as the supply curve intersects the horizontal axis at k = $\underline{a}$. Since the demand curve lies below the supply curve for small k, and above it for large k, then given the continuity of both curves, there must be at least one intersection. **Q.E.D.**

Corollary: If there is no capital saturation, there is an odd number of intersections : capital saturation is necessary for an even number of intersections.

However, not every intersection between the supply and demand curves constitutes an equilibrium in which the economy is sure to survive unfavorable production shocks. To insure the economy against vanishing due to lack of savings, capital stock in equilibrium must be such that

$$\varepsilon^* < f(k) - \underline{a} \qquad \text{(using (7))} \tag{8}$$

Thus equilibria where the capital stock is not large enough to satisfy (8) are not "safe": there is some danger of the economy disappearing.

Examples

1. Consider the standard Cobb-Douglas production function f(k) = k^α. Using (7), in equilibrium,

$$\alpha k^\alpha + k = \underline{a} \tag{9}$$

While the right hand side of (9) is invariant with k, the left hand side is monotonically increasing : its derivative being

$$1 + \alpha^2 k^{-(1-\alpha)} > 1 > 0.$$

Therefore, only one k satisfies (9). Moreover, there is no capital saturation : $f'(k) = \alpha k^{-(1-\alpha)} \to 0$ only as $k \to \infty$. Therefore, by Proposition 1, a unique equilibrium exists (**Figure 1**). Consider parameter values α = 1/3, $\underline{a}$ = 7/24, ε = 1/8. Then the equilibrium k satisfying (9) is k = 1/8. Moreover, (8) is satisfied, as its left hand side is 1/8 while its right hand side is 5/24 > 1/8. Thus in this equilibrium, there is no danger of not being able to save enough to meet the target.

2. Consider the piecewise quadratic production function

$$f(k) = ak - bk^2 \text{ for } k \le a/2b$$
$$= a^2/4b \text{ for } k \ge a/2b.$$

This production function displays capital saturation, a necessary condition for an even number of intersections. Here the demand curve is linear with a horizontal intercept at

[§] *Op.cit.*

a/2b and a vertical intercept at a (**Figure 2**). Since the horizontal intercept of the supply curve is $\underline{a}$, we see that if a/2b < $\underline{a}$, then provided the demand curve crosses the supply curve at least once, it will do so an even number of times – specifically, twice for this production function. Then, again using (7), any equilibria must satisfy

$$k + ak - 2bk^2 = \underline{a}$$

or

$$2bk^2 - (1+a)k + \underline{a} = 0 \tag{10}$$

The solutions to (10) are given by

$$k_1, k_2 = \frac{1 + a \pm \sqrt{(1+a)^2 - 8b\underline{a}}}{4b} \tag{11}$$

These roots are both real provided

$$(1+a)^2 > 8b\underline{a} \tag{12}$$

Moreover, for any a, b > 0, the sum and product of roots is positive, so that both roots are positive. For both roots to be less than the saturation level of capital (so that the interest rate is non-negative), a sufficient condition is that the larger root be less than a/2b. This reduces to

$$\frac{a}{2b} < \underline{a} \tag{13}$$

– the condition mentioned in the text for the horizontal intercept of the demand curve to be less than that of the supply curve (permitting two crossings). Consider parameter values of a = 8, b = 1, $\underline{a}$=7.5, $\varepsilon^* = .08$. Then we have $k_1 = 3.4$, $k_2 = 1.1$. These parameters satisfy both (12) and (13), which is why we have two real positive roots. Both roots are less than the value at which capital saturation sets in, which is 4 for these parameter values. Output at the first of these equilibria is $f(k_1) = 15.6$ while output in the second equilibrium is $f(k_2) = 7.59$. Moreover, (8) is satisfied at both these outputs so that even in the most unfavorable case, there is no danger of the economy disappearing. However, these two equilibria are clearly Pareto-rankable, as output in the first exceeds that in the second.

More generally, the equilibrium with the larger capital stock will involve larger output as marginal productivity of capital is positive (marginal productivity is equated to the prenegotiated interest rate, which can never be negative – or else people would prefer to store their savings instead of investing them in the production technology). Because savings are negatively interest-elastic, the equilibrium with the higher capital stock and higher output is associated with a low interest rate, while the other equilibrium is a high interest-low output equilibrium. The former type of equilibrium appears to describe the Japanese economy, where high savings have persistently coexisted with very low interest rates, though the levels of output and employment have been high. If indeed much of the population comprises target savers, a low-interest equilibrium is better, in terms of output than one with a higher interest rate – accordingly, attempts to move from one to the other may be misguided.

It is also possible to have multiple equilibria where the equilibrium with the smaller capital stock yields an output too low to satisfy (8). In this case, although the low output-high interest equilibrium can be maintained, given a run of good luck, a strong adverse shock can destroy the economy. In contrast, the high-output low-interest equilibrium could have been robust to strong unfavorable shocks. Therefore, if strong shocks are possible, it is even more desirable to co-ordinate on the high-capital stock equilibrium than on the Pareto inferior one. Whether government or any other institution can play such a co-ordinating role, aiding equilibrium selection, remains an open question.

4. Conclusion

We have analyzed an overlapping generations model of "target savers". With rational expectations, the older generation can insure themselves against all production shocks by setting a pre-negotiated interest rate on the savings they loan to the productive younger generation. The model exhibits excess sensitivity of consumption to transitory income. We have identified some conditions governing the existence of equilibrium, and the number of equilibria. We have given examples of a unique equilibrium and have also shown that multiple Pareto-rankable equilibria are possible. In the latter case we may have high-interest low-saving equilibria as well as low-interest high-saving equilibria : the latter seem to describe the situation of the Japanese economy. Co-ordinating on the Pareto superior equilibrium is even more important if the Pareto-inferior one is such that a particularly unfavorable shock may destroy the economy.

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Figure 1

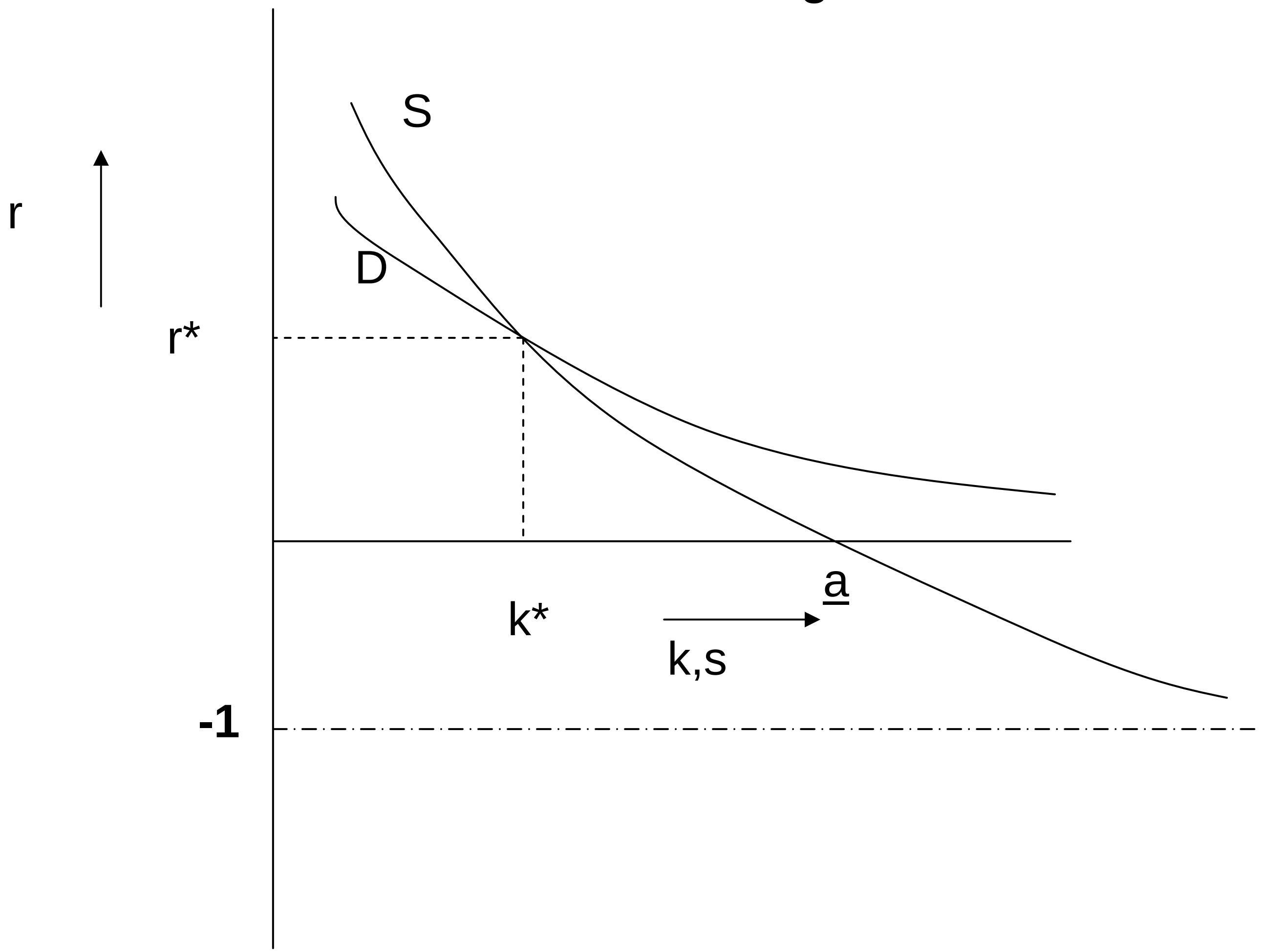

Figure 2