Incorporating Satisfaction into Customer Value Analysis: Optimal Investment in Life-Time Value

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This project is partly funded by the Wharton-SMU Research Center of Singapore Management University
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October 2003

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Abstract

We take Schmittlein, Morrison, and Colombo’s model (1987) of customer’s life-time value one step further to include customer satisfaction. The basic premise is that a customer purchases more frequently when she is happy than when she is disgruntled. We model customer purchase arrivals as Poisson events and allow the rate of arrival to depend on the satisfaction of the most recent service encounter. We derive a closed-form formula for predicting total number of purchases from a customer base over a time period \((0, T]\). This formula reveals that approximating the mixture arrival processes by a single “equivalent” Poisson process (independent of satisfaction) can systematically underestimate the total number of purchases. Since life-time value is an explicit function of customer satisfaction, we can use our model as a tool for quantifying the marginal return for investing in customer satisfaction. Since life-time value is convex in customer satisfaction, it is optimal to pursue 100% satisfaction if the cost is linear. We also characterize the conditions where it is not optimal for firms to pursue customer satisfaction relentlessly.

Keywords: Customer Satisfaction, Life-time Value Analysis, Customer Relationship Management
1 Introduction

Customers are assets and their values grow and decline. Predicting and managing customers life-time value is central to marketing because the health of a firm is intimately related to the health of its customer base. Segmenting customers based on their life-time value is a powerful way to target them. This paper provides an analytical framework for predicting and managing a customer’s life-time value based on her satisfaction with the firm.

The importance of this issue is quite evident in the burgeoning practitioner literature on customer relationship management (CRM). Industry experts have emphasized the importance of incorporating satisfaction metrics into customer valuation and advise firms to balance customer satisfaction and cost control (e.g., Forrester Research 2003; Jupiter Research 2000). Our research provides a formal way to assess investment in customer satisfaction by linking it to likely future behavioral shopping patterns and hence revenue flow.

Our modeling framework rests on two premises. First, we posit that customer satisfaction is a key determinant of life-time value. That is, *ceteris paribus*, a customer will purchase more frequently when she is happy (i.e. had a positive previous purchase encounter) than when she is unhappy (i.e. had a negative previous purchase encounter) with the firm. This premise is both intuitively appealing and empirically sound. Several studies have shown that customer satisfaction is a good predictor for likelihood of repeat purchase and revenue growth (e.g., Anderson and Sullivan 1993; Jones and Sasser 1995). In addition, prior research has shown that customers react negatively to poor service (e.g., stockouts) by switching to another firm on subsequent shopping trips, which may result in a significant reduction on future demand (Fitzsimons 2000; Anderson, Fitzsimons, and Simester 2002).

Second, it is feasible to invest in costly productive processes to increase customer satisfaction. For instance, a call center that increases its number of customer representatives will reduce queuing time. Similarly, a catalog firm can improve its logistics processes to shorten delivery time and reduce the incidence of wrong shipments. The investment of these costly productive processes, however, requires a formal quantification of their revenue implication. A goal of this research is to derive an analytical relationship between revenue and customer satisfaction by
developing a micro-level customer purchase model.

We build on the seminal work of Schmittlein, Morrison, and Colombo (hereafter, abbreviated as SMC) (1987) and Schmittlein and Peterson (1994). Their model assumes that customer purchase arrivals are Poisson events. Customers are allowed to die (i.e., switch to another firm or leave the product category entirely) in a Poisson manner so that the number of active customers can decline over time. Customers are heterogeneous in their purchase intensity and death propensity. The heterogeneities in both the purchase and death rates are modeled by two independent gamma distributions. The amount spent on each purchase is normally distributed and is assumed to be independent of the arrival and death processes. They derive an elegant formula to predict the total number of purchases from a customer base over a time period. There are at least four ways customer satisfaction can affect this classical customer purchase model:

1. Higher Arrival Rate: A happy customer is likely to make more trips to the firm. In other words, the firm can increase its share of pocket revenue of the product category by making the customer happy.

2. Lower Death Rate: A happy customer is less likely to switch to another seller or leave the product category entirely.

3. Higher Expenditure: A happy customer may increase her spending in the product category for each visit.

4. Positive Word-of-Mouth: A happy customer may spread positive word-of-mouth to other potential customers and hence increase the number of new customers.

In this paper, we focus on the effect of customer satisfaction on arrival rate of purchase and expenditure. A comprehensive modeling of #2 requires extending the proposed model framework to a competitive context. We leave word-of-mouth effect for future research. Examples of the proposed modeling situations include the following:

- A customer who visited a local retailer (e.g., restaurant, supermarket, gas station, etc.) and had a good experience is more likely to return to the retailer sooner.
• A disgruntled manufacturer who sourced from a supplier who missed delivery schedule may downgrade the supplier and purchase less frequently from that particular supplier in the future.

• An unhappy customer who received a catalog order late or a wrong shipment may subsequently purchase less frequently from that catalog company.

• A TV audience is more likely to follow a weekly program if she was happy with it in the prior week. Similarly, a second-year MBA student who just had a good class session is less likely to miss the next class session.

This paper makes three main contributions:

1. We extend Schmittlein, Morrison, and Colombo’s framework to include satisfaction in predicting customer’s life-time value. We derive a formula to predict the total number of expected purchases from a customer base. This formula allows the firm to predict life-time value based on customer satisfaction, a key indicator of customer health.

2. We show the total number of purchases is convex and increasing in customer satisfaction. This somewhat surprising result may explain why some firms relentlessly pursue customer satisfaction in practice. In addition, we find that one will underestimate the total number of purchases if multiple satisfaction levels are treated as a single “equivalent” aggregate level, and that this systematic bias peaks at a customer satisfaction of 50%.

3. Our framework can be used as a tool for quantifying the marginal benefit of improving customer satisfaction. Consequently, the framework allows the firm to actively manage its productive processes to increase customer’s life-time value. Using the framework, we show when it is not optimal for the firm to pursue customer satisfaction at all costs.

The remainder of the paper is organized as follows. Section 2 presents the model formulation and characterizes the total number of expected purchases and the optimal level of customer satisfaction. Section 3 provides numerical examples to demonstrate the main theoretical results.
Section 4 presents two extensions to the proposed model. Section 5 concludes and suggests future research directions.

2 Model Development

We consider a firm that offers a homogeneous product or service to all its customers. Since the production process is inherently stochastic, a customer is satisfied with probability $p$, and dissatisfied with probability $1 - p$ at each purchase encounter. We assume that the production process does not discriminate customers and that it is independent of previous purchase encounters so that the service outcomes can be modeled as independent and identically distributed binomial trials.

The rate of arrival of a customer is assumed to follow a Poisson process whose rate changes with the outcome of each service encounter. When a customer is dissatisfied, her next purchase comes with arrival rate $\lambda_1$; when a customer is satisfied, the arrival rate for next purchase is $\lambda_2$. Note that $\lambda_2 > \lambda_1$. We assume a Markovian property such that the arrival rate depends only on the most recent service encounter. This assumption is reasonable if the inter-purchase time is long enough or the customer exhibits a kind of “recency effect” and reacts strongly to the most recent service encounter. In addition, each customer has a defection rate of $\mu$. If a customer has defected, she is “dead”; otherwise, the customer is “alive”.

Suppose we are currently at time 0. We are interested in addressing the following questions:

- What is the total number of expected purchases a given customer is expected to make during $(0, T]$?

- What is the total number of expected purchases a customer base is expected to make during $(0, T]$ if arrival and death rates differ across customers?

---

2Our model can be extended to investigate more than two levels of satisfaction (e.g., three levels such as below expectation, meets expectation, exceeds expectation). For ease of exposition, we restrict ourselves to dichotomous levels such as happy versus unhappy, exceeds expectation versus below expectation, satisfied versus dissatisfied, and so on.
Given a cost of providing customer satisfaction $p$, what’s the optimal customer satisfaction $p^*$ that maximizes customer life-time value?

We will address these questions one by one in the following sections.

2.1 Total Expected Purchases During $(0, T]$

Since the purchase arrival rate of each customer depends on the outcome of the previous service encounter, the inter-arrival time is a hyper-exponential random variable: with probability $1 - p$ it is exponential with rate $\lambda_1$; and with probability $p$ it is exponential with rate $\lambda_2$. For ease of exposition, we make two simplistic assumptions, both of which will be relaxed later.

1. Each time a customer visits a store, the amount of money she spends is independent of the outcome of the service (i.e. whether she is happy or unhappy). Rather, it follows a random distribution with expectation $Q$. Because the amount spent is independent of the arrival and death processes, it is easy to see that the total expected purchase amount is simply the product of $Q$ and the expected number of total purchase visits. In section 4.2, we will let $Q$ depend on the service outcome: a customer spends more money when she is happy than when she is unhappy during the next purchase encounter.

2. We assume a zero discount rate. In section 4, we extend this model by factoring the discount rate into the calculation of the net present value of the expected purchases.

Clearly, the total expected number of purchases during $(0, T]$ depends on whether the customer is happy or unhappy at time $t = 0$. The following theorem provides the expressions:

**Theorem 1**

Let $\lambda = p\lambda_1 + (1 - p)\lambda_2$.

If a customer is unhappy at time 0, the expected total purchase during $(0, T]$ is:

$$\pi_u = Q \left[ \frac{\lambda_1 \lambda_2}{\lambda \mu} \left(1 - e^{-\mu T}\right) - \frac{p\lambda_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda + p)T}\right) \right].$$ (2.1)
If a customer is happy at time 0, the expected total purchase during \((0, T]\) is:

\[
\pi_h = Q \left[ \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T}) + \frac{(1 - p) \lambda_2 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda + \mu) T}\right) \right].
\]  

(2.2)

The expected total purchase is:

\[
\pi = p \pi_h + (1 - p) \pi_u = Q \left[ \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T}) + \frac{p(1 - p) (\lambda_2 - \lambda_1)^2}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda + \mu) T}\right) \right].
\]  

(2.3)

**Proof**

See Appendix A.

Note that \(\pi_h > \pi_u\) so that customer purchases more when she is initially happy. Let \(\Delta \pi_{hu} = \pi_h - \pi_u = \frac{\lambda_2 - \lambda_1}{\lambda \mu} \cdot (1 - e^{-(\lambda+\mu) T})\). It can be easily shown that \(\Delta \pi_{hu}\) increases the difference in arrival rates \((\lambda_2 - \lambda_1)\) and decreases the death rate \(\mu\). Thus, satisfaction is more important in markets where customers have a longer expected life.

The overall expected purchase \(\pi\) increases and convex in \(p\) (see Appendix C.1 for a proof). In other words, the total expected purchase has an increasing return to scale in customer satisfaction. This is a surprising but powerful result. This conclusion provides a formal justification of why many firms invest at any cost in customer satisfaction. Our result suggests that this is optimal as long as costs are linear in customer satisfaction.

If the customer satisfaction level is not differentiated, neither will the purchase arrival rate. The equivalent “uniform” purchase arrival rate in that case will be \(\lambda_s\) such that:

\[
\frac{1 - p}{\lambda_1} + \frac{p}{\lambda_2} = \frac{1}{\lambda_s} \quad \Rightarrow \quad \lambda_s = \frac{\lambda_1 \lambda_2}{(1 - p) \lambda_2 + p \lambda_1} = \frac{\lambda_1 \lambda_2}{\lambda}.
\]  

(2.4)

This expression is based on the assumption that the average inter-purchase time under our proposed model is equal to that under the SMC model. Based on this expression, the first term in (2.3) can be written as \(\frac{\lambda_2 - \lambda_1}{\lambda \mu} \left(1 - e^{-\mu T}\right)\), which is the total expected purchases given in Schmittlein *et al.* (1987). The second term in (2.3) represents the difference in estimating total expected purchases between our framework and theirs. Clearly, when \(p = 1\) or \(\lambda = \lambda_2\), this difference vanishes. The formula implies that if we ignore a customer’s reaction to service quality, and approximate the underlying mixture Poisson processes by a single “equivalent” Poisson process, we will underestimate the expected number of purchases by \(\frac{p(1 - p)(\lambda_2 - \lambda_1)^2}{\lambda (\lambda + \mu)} (1 - \ldots}
$e^{-(\lambda+\mu)T}$. This bias has an inverse U-shape in customer satisfaction and peaks when $p = 0.5$. It also increases with the difference in arrival rates.

2.2 Customer Heterogeneity

We now turn our attention to incorporate customer heterogeneity on the purchase rates and death rates across the population. In order to do so, we assume that a customer who has a higher $\lambda_1$ also has a higher $\lambda_2$. In other words, there is a perfect correlation between the two arrival rates, (i.e., $\lambda_1 = k\lambda_2$ ($0 < k < 1$) and thus $\lambda = [1 - (1 - k)p]\lambda_2$). Let $p_k = [1 - (1 - k)p]$ so that $\lambda = p_k\lambda_2$. This assumption implies that a customer who is overall a bigger spender will buy more than a thrifty shopper not only when she is happy but also when she is unhappy, reflecting that the customer has an inherent demand for the product or service regardless of her level of satisfaction with the firm. A more general model can allow $\lambda_1$ and $\lambda_2$ to correlate but not perfectly and we shall leave this for future research.

We assume the independent gamma mixing distributions with parameters $(r; \alpha)$ on $\lambda_2$ and $(s; \beta)$ on $\mu$:

$$g_{\lambda_2}(\lambda_2; r, \alpha) = \frac{\lambda_2^{r-1}e^{-\alpha\lambda_2}}{\Gamma(r)}$$

and

$$g_{\mu}(\mu; s, \beta) = \frac{\mu^{s-1}\beta\mu e^{-\beta\mu}}{\Gamma(s)},$$

and $E[\lambda_2|r, \alpha] = \frac{r}{\alpha}$; $\text{Var}[\lambda_2|r, \alpha] = \frac{r}{\alpha^2}$, and $E[\mu|s, \beta] = \frac{s}{\beta}$; $\text{Var}[\mu|s, \beta] = \frac{s}{\beta^2}$. Thus, the coefficient of variation (i.e., standard deviation divided by the mean) of $\lambda_2$ is $r^{-1/2}$. $r$ is thus an index of homogeneity in purchase rates over the customer base. In the same way, $s$ indicates the homogeneity in death rates.

Note that each latent rate has completely different (independent) parameter sets. We integrate (2.1) for customers starting unhappy and (2.2) for customers starting happy across all possible values of $\lambda_2$ and $\mu$ using the corresponding gamma mixing distributions.

Theorem 2

Case 1: $p_k \neq \frac{2}{3}$. 

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If a customer is unhappy at time 0, the expected total purchase during \([0,T]\) is:

\[
\pi_u = Q \times \left\{ - \frac{kr\beta}{pk\alpha(s-1)} \left[ 1 - \left( \frac{\beta}{\beta+T} \right)^{s-1} \right] + \frac{pk(1-k)\alpha^r\beta^s}{pk^2} \left[ \frac{F(a,b;c;z_1)}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} - \frac{F(a,b;c;z_2)}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} \right] \right\}, \tag{2.5}
\]

where \(a = r + s, b = r + 1, c = r + s + 1, z_1 = \frac{pk\alpha - \beta}{pk\beta}, \) and \(z_2 = \frac{pk\beta - \alpha}{pk(\beta+T)}\).

If a customer is happy at time 0, the expected total purchase during \([0,T]\) is:

\[
\pi_h = Q \times \left\{ \frac{kr\beta}{pk\alpha(s-1)} \left[ 1 - \left( \frac{\beta}{\beta+T} \right)^{s-1} \right] + \frac{(1-p)(1-k)^2\alpha^r\beta^s}{pk^2} \left[ \frac{F(a,b;c;z_1)}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} - \frac{F(a,b;c;z_2)}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} \right] \right\}. \tag{2.6}
\]

The total expected purchase is \(\pi = p\pi_h + (1-p)\pi_u\):

\[
\pi = Q \times \left\{ \frac{kr\beta}{pk\alpha(s-1)} \left[ 1 - \left( \frac{\beta}{\beta+T} \right)^{s-1} \right] + \frac{(1-p)(1-k)^2\alpha^r\beta^s}{pk^2} \left[ \frac{F(a,b;c;z_1)}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} - \frac{F(a,b;c;z_2)}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} \right] \right\}. \tag{2.7}
\]

Case 2: \(p_k = \frac{\alpha}{\beta}\).

If a customer is unhappy at time 0, the expected total purchase during \([0,T]\) is:

\[
\pi_u = Q \times \left\{ - \frac{kr\beta}{pk\alpha(s-1)} \left[ 1 - \left( \frac{\beta}{\beta+T} \right)^{s-1} \right] \right\}. \tag{2.8}
\]

If a customer is happy at time 0, the expected total purchase during \([0,T]\) is:

\[
\pi_h = Q \times \left\{ \frac{kr\beta}{pk\alpha(s-1)} \left[ 1 - \left( \frac{\beta}{\beta+T} \right)^{s-1} \right] + \frac{(1-p)(1-k)^2\alpha^r\beta^s}{pk^2} \left[ \frac{1}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} - \frac{1}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} \right] \right\}. \tag{2.9}
\]

The total expected purchase is \(\pi = p\pi_h + (1-p)\pi_u\):

\[
\pi = Q \times \left\{ \frac{kr\beta}{pk\alpha(s-1)} \left[ 1 - \left( \frac{\beta}{\beta+T} \right)^{s-1} \right] + \frac{(1-p)(1-k)^2\alpha^r\beta^s}{pk^2} \left[ \frac{1}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} - \frac{1}{1 - \left( \frac{\beta}{\beta+T} \right)^{s-1}} \right] \right\}. \tag{2.10}
\]

Proof: See Appendix B.

In (2.5)–(2.7), \(F(a,b;c;z)\) is the Gauss hypergeometric function (Abramowitz and Stegun 1972, p. 558).

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When the customer satisfaction level is not differentiated, as similar to (2.4) in the previous section, the equivalent “uniform” gamma parameter in the arrival rate is $r_s$ such that:

$$
r_s = \left( \frac{k}{1 - p + kp} \right) r = \left( \frac{k}{pk} \right) r.
$$

(2.11)

This expression will be used in the numerical examples to show the difference between our framework (which explicitly considers service quality) and SMCs framework (which does not consider service quality). Based on the above expression, the first term in (2.7) and (2.10) can be written as:

$$
\pi = \frac{r_s \beta}{\alpha(s - 1)} \left[ 1 - \left( \frac{\beta}{\beta + T} \right)^{s-1} \right],
$$

(2.12)

which is the expected number of purchases given in Schmittlein et al. (1987). The second term in (2.7) and (2.10) represents the difference in the expected number of purchases between these two models. As a result, the qualitative nature of the previous sections results remains unchanged.

### 2.3 Optimal Investment

It is clear that the total revenue as a function of customer satisfaction $p$ is increasing. This does not mean that the firm should pursue customer satisfaction 100%, because there may be prohibitive investment costs associated with this pursuit. In this section, we develop the net profit as a function of customer satisfaction in order to determine the optimal investment in customer satisfaction. Since we have already shown (see Appendices C.1 and C.2) that the revenue function has increasing returns in service quality (i.e., it’s convex), the optimal level of investment in service quality will depend on the shape of the cost function. It would be hard to find cases where the cost as a function of service quality decreases. Therefore, we will analyze only increasing cost functions. Consider the following scenarios:

- The investment cost is linear in service quality level. In this case, as the company invests more in service quality, it gets more in revenue return (due to the revenue function’s convexity in $p$), but incurs a constant marginal cost. Therefore, the higher the quality...
level, the better the profit. It makes sense for companies to want to achieve a 100% service quality. This is evidenced by many companies’ emphasis on service quality and their pursuit to achieve perfect service quality.

- The investment cost is concave in service quality level. For example, the cost function may have a fixed cost and a linear variable cost component. In this case, as the company invests more to increase service quality, the revenue return increases while the marginal cost decreases or remains constant. So clearly, just as in the previous case, it is optimal for the company to pursue a perfect service quality.

- The investment cost is convex in service quality level. That is, it costs money to improve customer satisfaction, and it is increasingly harder to do so. More often than not, this is the case in reality. It is not immediately clear what the profit looks like as a function of $p$. Intuitively, when the cost function is “less convex” than the revenue function (i.e., both marginal revenue and marginal cost are increasing in $p$, but the former outpaces the latter), then it again makes sense to pursue a perfect service quality. If the cost function is “more convex” than the revenue function, however, the profit function will eventually decrease as service quality becomes higher and higher. This means that an interior optimal point exists for $p$. That is, it is best to invest in service quality up to a level less than 100%.

In this section, we will analyze the third case (convex cost function). The cost function we analyze is in the form $M(1 - p)^{-\omega}$. So the parameter $\omega$ represents the “convexness” of the cost function (the higher the $\omega$, the more convex the cost function). We will show that, under reasonable conditions, i.e. sufficiently big $\omega$ and sufficiently small $M$, there exists an interior optimal customer satisfaction (see Theorem 3).

Examples of convex cost function in $p$ abound in service systems:

- In an $M/M/c$ queueing service system (to conform to conventional queueing notation, we will re-use the notations $\lambda$ and $\mu$ to denote the arrival rate and service rate), if the cost is directly proportional to either individual $\mu$ or the number of servers $c$, and service quality
is measured as the probability of not having to wait in queue at all, \( p_0 \), or the average waiting time in the system, then the cost function is convex (for details, see Kleinrock 1975).

- In a \( M/M/c/K \) finite-waiting-space queueing service system, if the service quality is measured by loss rate and the cost function is directly proportional to either individual \( \mu \) or the number of servers \( c \), then the cost function is convex in service quality.

- In the well-known single-period newsboy inventory model, demand is unknown with a given CDF \( F(\cdot) \). The service level is usually defined as the probability that the customer demand is satisfied. Therefore, if the firm carries \( x \) units of inventory, the service level it provides is \( F(x) \). To achieve such service level, the firm incurs an inventory holding cost of \( hx \) where \( h \) is the inventory holding cost per unit inventory per unit time. Clearly, if the service level is equivalent to service quality \( p \), and the inventory holding cost is the investment necessary to achieve such service quality, then the cost function is \( h F^{-1}(p) \).

As long as \( F^{-1} \) is convex in \( p \) (or, \( F \) is concave in \( x \)), the cost function is convex. Any distribution that has a monotonically non-increasing pdf satisfies such a condition. For example, exponential distribution, certain Weibull and Gamma distributions, and uniform distribution all have monotonically non-increasing pdf.

The conditions on \( \omega \) and \( M \) make sense. On the one hand, the cost function approaches infinity when \( p \to 1 \); on the other hand, revenue function will approach a finite number as \( p \to 1 \). If the cost function is “more convex” than the revenue function, then the profit function is concave and has a unique maximum. This is ensured by a large \( \omega \) because \( \omega \) solely determines the convexness of the cost function. A small \( M \) ensures positive profit at small values of \( p \).

Mathematically, if the customer base size is \( N \), the profit is calculated as:

\[
\pi_a = N \cdot Q \cdot \int \pi d \Sigma(\lambda_1, \lambda_2, \mu) - C(p),
\]

(2.13)

where \( \Sigma(\lambda_1, \lambda_2, \mu) \) is a probability distribution on \( (\lambda_1, \lambda_2, \mu) \), and \( C(p) = M(1 - p)^{-\omega} \) is the cost function.
We now analyze the revenue function, the cost function, and the profit function, respectively. Recall the revenue function shown in (2.3). Consequently, we have $\frac{d\pi}{dp} \geq 0$, $\frac{d^2\pi}{dp^2} \geq 0$, and $\frac{d^3\pi}{dp^3} \geq 0$. (see Appendix C.1 for proof). The cost function has the following properties:

\[
C(p) = M(1-p)^{-\omega}; \quad C(p)|_{p=0} = M
\]
\[
C'(p) = M\omega(1-p)^{-(\omega+1)}; \quad C'(p)|_{p=0} = M\omega
\]
\[
C''(p) = M\omega(\omega + 1)(1-p)^{-(\omega+2)}; \quad C''(p)|_{p=0} = M\omega(\omega + 1)
\]

We will show that under general and reasonable conditions, there exists a unique investment level in service quality that maximizes profit. It is sufficient for the profit function, $\pi_a$ as defined in (2.13), to satisfy the following conditions:

1. $\partial^2 \pi_a(p)/\partial p^2 \leq 0$ (concavity);
2. $\partial \pi_a(p)/\partial p$ is positive at $p = 0$ and negative at $p = 1$ (internal maximum); and
3. $\pi_a(p)$ is positive at some point $p \in (0, 1]$ (positive profits at the optimal point).

Because both $\pi$ and $C$ are convex functions, for $\pi_a$ to be a concave function, we must have $C$ “more convex” than $\pi$. Since the “convexness” of $C$ depends on $\omega$, it makes sense that $\omega$ should be large. Moreover, to have a positive value for $\pi_a$ at small values of $p$, it again makes sense that $M$ should be small. Indeed it can be shown that when $\omega$ is big enough and $M$ is small enough, all three conditions are met. Theorem 3 states these conditions formally:

**Theorem 3**  
(i) There exists finite $B_1$, $B_2$, and $B_3$ (which may depend on $\lambda_1, \lambda_2$, and $\mu$) such that $B_1 \leq \pi'(p) \leq B_2$ and $\pi''(p) \leq B_3$ for all $p \in (0, 1]$. Moreover, $\int B_1 d\Sigma$, $\int B_2 d\Sigma$, and $\int B_3 d\Sigma$ are finite.

(ii) If $\omega$ and $M$ are such that $\omega M < NQ \int B_1 d\Sigma$, $\omega(\omega + 1)M > NQ \int B_3 d\Sigma$, and $M < NQ\pi(0)$, then conditions 1-3 are satisfied, and there exists a unique interior optimal $p$ that maximizes $\pi_a$.

**Proof**  See Appendix C.3.  

\[\square\]
3 Numerical Examples

At the heart of our model lies two different purchase arrival rates: one for happy customers and one for unhappy customers. As discussed in section 2, failure to account for heterogeneous purchase rates for happy and unhappy customers may lead to a systematic bias in estimating the expected number of purchases and hence customer life-time value. To illustrate this point more clearly, we present numerical results in which we compare the SMC model to our proposed model. The purpose of this numerical analysis is to identify and quantify the nature and magnitude of the potential biases caused by ignoring heterogeneous purchase rates due to changes in customer satisfaction.

We compute the expected number of purchases and characterize the optimal investment for the customer satisfaction based on the four parameters in the two gamma mixing distributions. This analysis is particularly relevant for situations where purchase rates could be very different depending on the previous service encounter. In order to compute the expected number of purchases, we use equation (2.12) for the SMC model and Theorem 2 for our model. To define the optimal level of customer satisfaction, we employ equation (2.13) as well as expressions for the expected number of purchases.

We choose the following four parameters in the proposed model for the purchase/death process: \( r = 0.415, \alpha = 0.415, s = 0.300, \beta = 0.600 \). Thus, the average purchase rate is \( r/\alpha = 1.0 \) and the average death rate is \( s/\beta = 0.5 \) unit per time period. The relatively small \( r \)-value of 0.415 indicates substantial differences in purchase rates across customers. In the same way, the relatively small value of 0.300 for \( s \) indicates that the death rate also varies greatly from customer to customer. We chose these values to be consistent with previous empirical estimates (e.g., Morrison and Schmittlein 1981; Schmittlein, Morrison and Colombo 1987) and were used throughout the numerical examples in this section.

We next define the four parameters in the SMC model. In order to characterize the gamma parameters for the purchase process, we let \( \lambda_1 = k\lambda_2 \) \((0 < k < 1)\) and assume that the average inter-purchase time under our model is equal to that under the SMC model. The expression shown in (2.11) results, where the same scale parameter is implicitly assumed (i.e., \( \alpha_s = \alpha \)).
In addition, our model assumes the same death process as the SMC model: \( s_s = s \) and \( \beta_s = \beta \), where the subscript \( s \) stands for the SMC model. Besides the parameters for the two independent gamma distributions, both \( T \) and \( Q \) are set to 1 for the numerical illustrations.

The results of the expected number of purchases reveal the importance of accounting for heterogeneous purchase rates in the customer value analysis. Figure 1 illustrates that the SMC model underestimates the expected purchases. The magnitude of the bias is directly related to the value of \( k \) which indicates the degree of heterogeneity in purchase arrival rates. If \( k \) is close to 1, then we wouldn’t observe any difference between these two models. In addition, the amount of the bias is larger when \( p \) becomes smaller.

**Insert Figure 1 about here**

We investigate optimal customer satisfaction and its corresponding profit by incorporating cost investments. Besides all the parameters in the previous example, we need to set parameters for cost functions \( (M \) and \( \omega) \) as well as for the customer base size \( (N) \) in order to characterize the optimal investment of customer satisfaction. We set \( N \) equal to 1000 and choose a set of parameter values for \( M \) and \( \omega \) in cost function for illustration purposes. Note that \( M \) can be interpreted as the initial cost while \( \omega \) can be interpreted as the marginal cost to improve customer satisfaction. Finally, we set \( k \) equal to 0.50 in the following examples.

In Figures 2 and 3, we fix \( \omega \) equal to 3.0, but vary the value of \( M \), ranging from 0.001 to 0.500. Figures 2 and 3 show optimal customer satisfaction and its corresponding profits, respectively. As shown in these figures, optimal levels of customer satisfaction and its corresponding profits decrease as \( M \) increases.

**Insert Figures 2 and 3 about here**

In Figures 4 and 5, we fix \( M \) equal to 0.001, but vary the value of \( \omega \), ranging from 1.0 to 10.0. Figures 4 and 5 show optimal customer satisfaction and its corresponding profits, respectively. As shown in these figures, optimal levels of customer satisfaction and its corresponding profits decrease as \( \omega \) increases.

**Insert Figures 4 and 5 about here**
Looking at the magnitudes of optimal customer satisfaction, it appears that the overall importance of cost investments is moderately higher in $\omega$ than in $M$. In sum, Figures 2–5 illustrate that optimal customer satisfaction and profits decrease with $M$ and $\omega$. That is, as it becomes more costly to invest in productive processes to increase customer satisfaction, a firm should invest less.

4 Model Extensions

In this section, we present two extensions of the proposed model: (1) a discounting framework for the total expected purchases and (2) contingent purchase amount based on customer satisfaction.

4.1 Discounting Framework

When the time frame under consideration is considerably long, studying the net present value (NPV) of a customer’s purchases makes more sense, than studying the simple sum in the previous section. There are two ways to consider the discounting:

1. Continuous discount: A purchase of 1 made at time $t$ has a NPV of $e^{-\gamma t}$, with $\gamma$ being the discount factor;

2. Discrete-continuous discount: Time $(0, T]$ is broken into $K$ equal-length intervals of length $T_0$ where $T = KT_0$. Purchases made in each interval are simply totaled, but the total purchases made in an interval is discounted to time 0 by using a discount factor of $d < 1$.

While continuous discounting is usually the more natural choice, it results in analytic intractability, because the arrival rate depends on the outcome of the last encounter. Instead, we use the discrete-continuous discounting scheme because it is analytically tractable. When we make $T_0$ small enough, the results approximate those of the continuous discounting. In the limit, when $T_0$ approaches 0, our results converge with those of the continuous discounting. Moreover, the discrete-continuous discounting framework is realistic and practical. For technical and managerial reasons, most firms can not discount income on a continuous basis. Instead, firms may
accrue all the revenue during a day/week/month, and deposit the total only at the end of that time period.

The NPV of a customer’s total expected purchases during \((0, T]\) can be calculated in a similar way to (2.1)–(2.3):

\[ H = \sum_{i=0}^{K-1} (e^{-\mu T_0})^i \quad \text{and} \quad G = \sum_{i=0}^{K-1} (e^{-(\lambda+\mu)T_0})^i. \]

Then, if a customer is unhappy at time 0, the total expected purchase during \((0, T]\) is:

\[
\pi_u = Q \left[ H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T_0}) - G \frac{p \lambda_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda+\mu)T_0}\right) \right]. \tag{4.1}
\]

If a customer is happy at time 0, the expected total purchase during \((0, T]\) is:

\[
\pi_h = Q \left[ H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T_0}) + G \frac{(1-p) \lambda_2 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda+\mu)T_0}\right) \right]. \tag{4.2}
\]

The total expected NPV of a random customer is:

\[
\pi = p\pi_h + (1-p)\pi_u = Q \left[ H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T_0}) + G \frac{p(1-p)(\lambda_2 - \lambda_1)^2}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda+\mu)T_0}\right) \right]. \tag{4.3}
\]

Proof See Appendix C.4.

The following observations merit explicit attention. First, \(\pi\) is increasing and convex in \(p\) so that a high customer satisfaction always leads to a higher number of expected purchases. Second, the NPV now depends on \(K\) only through \(G\) and \(H\). In particular, if we let \(K = 1\) (i.e., \(T_0 = T\)), \(H = G = 1\) and (4.1) and (4.2) simplify to (2.1) and (2.2), respectively. If we let \(d = 1\), we have:

\[
H = \frac{1 - e^{-\mu T}}{1 - e^{-\mu T_0}} \quad \text{and} \quad G = \frac{1 - e^{-(\lambda+\mu)T}}{1 - e^{-(\lambda+\mu)T_0}},
\]

and (4.1) and (4.2) become (2.1) and (2.2), respectively. Finally, the NPV expression based on the SMC framework would be the first term in (4.3). The second term in (4.3) would represent the difference in NPV between our model and the SMC model. Note that because \(H > G\), the relative difference between these two models decreases because of the discount factor. In summary, the qualitative nature of the results discussed under the zero discount framework remains unchanged.
Based on the expressions in Theorem 4, we next capture customer heterogeneity on the purchase and death rates across the customer base. Similar to section 2.2, we integrate (4.1) for customers starting unhappy and (4.2) for customers starting happy across all possible values of $\lambda_2$ and $\mu$ using the corresponding gamma mixing distributions.3

**Theorem 5**

**Case 1:** $p_k \neq \frac{a}{b}$.

If a customer is unhappy at time 0, the expected total purchase during $[0, T]$ is:

$$\pi_u = Q \times \left\{ -\frac{kr\beta}{p_k\alpha(s-1)} \sum_{i=0}^{K-1} d^i \left[ \left( \frac{\beta}{\beta+i\cdot T_0} \right)^{s-1} - \left( \frac{\beta}{\beta+(i+1)T_0} \right)^{s-1} \right] \right\},$$  

where $a = r + s$, $b = r + 1$, $c = r + s + 1$, $z_1(i) = \frac{p_k\beta-\alpha}{p_k(\beta+iT_0)}$, and $z_2(i) = \frac{p_k\beta-\alpha}{p_k(\beta+(i+1)T_0)}$.

If a customer is happy at time 0, the expected total purchase during $[0, T]$ is:

$$\pi_h = Q \times \left\{ -\frac{kr\beta}{p_k\alpha(s-1)} \sum_{i=0}^{K-1} d^i \left[ \left( \frac{\beta}{\beta+i\cdot T_0} \right)^{s-1} - \left( \frac{\beta}{\beta+(i+1)T_0} \right)^{s-1} \right] \right\}.$$

**Case 2:** $p_k = \frac{a}{b}$.

If a customer is unhappy at time 0, the expected total purchase during $[0, T]$ is:

$$\pi_u = Q \times \left\{ -\frac{kr\beta}{p_k\alpha(s-1)} \sum_{i=0}^{K-1} d^i \left[ \left( \frac{1}{\beta+i\cdot T_0} \right)^{s-1} - \left( \frac{1}{\beta+(i+1)T_0} \right)^{s-1} \right] \right\}.$$

If a customer is happy at time 0, the expected total purchase during $[0, T]$ is:

$$\pi_h = Q \times \left\{ -\frac{kr\beta}{p_k\alpha(s-1)} \sum_{i=0}^{K-1} d^i \left[ \left( \frac{1}{\beta+i\cdot T_0} \right)^{s-1} - \left( \frac{1}{\beta+(i+1)T_0} \right)^{s-1} \right] \right\}.$$

In both cases, the total expected NPV of a random customer is $\pi = p\pi_h + (1-p)\pi_u$.

**Proof** See Appendix B.

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3We can generalize this integration to allow for the inter-dependence between the purchase and death rates. In order to complete this integration, we employ a bivariate gamma density of the Sarmanov family (Kotz, Balakrishnan, and Johnson 2000; Park and Fader 2003). Details are available upon request from the authors.
We have extended our framework to incorporate discounting in the revenue calculation. It is therefore natural to question whether the optimal investment in service quality will change as a result of revenue discounting. The following theorem addresses this question: Under similar conditions to that of Theorem 3, the optimal investment level continues to be an interior solution.

**Theorem 6** Theorem 3 continues to hold true for the NPV revenue function. Moreover, the optimal investment in service quality is lower with discounting than without.

To prove Theorem 6, we need to show that the first three derivatives of $\pi$ with respect to $p$ are all non-negative: $\pi'(p) \geq 0$, $\pi''(p) \geq 0$, and $\pi'''(p) \geq 0$. This is done in Appendix C.2. In Appendix C.3, we formally prove this Theorem 6.

The second statement of the theorem (that the optimal investment in service quality is lower with discounting) is intuitive: When the optimal service quality is an interior point, we always have “marginal revenue = marginal cost” at the optimal point. Revenue discounting, however, effectively reduces the marginal revenue at all the points (see (4.3)), while the marginal cost remains the same. Consequently, at the optimal point without discounting, the marginal revenue is smaller than the marginal cost, when discounting is considered. Therefore, the optimal service quality level should be lower as discounting is considered. It is also clear that the bigger the discount factor, the lower the optimal service quality level.

This result suggests that firms with a longer planning horizon tend to achieve a higher level of customer satisfaction. This may partially explain why Japanese firms tend to care more about customer satisfaction than US firms within the same industry because managers in the former are normally characterized as being more long-term oriented than their US counterparts.

### 4.2 Contingent Spending Amount

We assume that the amount spent on each purchase visit is independent of customer satisfaction. In this section, we analyze the dependent case. Specifically, when a customer is happy, she will spend a random amount with an average of $Q_h$; and when a customer is unhappy, she will
spend a random amount with an average of $Q_u$. It is natural to have $Q_h \geq Q_u$. We can show that the total amount this customer is expected to spend is simply $Q_h$ times the number of happy visits, plus $Q_u$ times the number of unhappy visits. Because a customer’s satisfaction at each visit is independent and identically distributed, the number of happy visits is expected to be $p$ fraction of the total visits and the number of unhappy visits is expected to be $1 - p$ fraction of the total visits. In summary, a customer’s total expected purchase is the expected total number of visits times $Q$, the overall average amount a customer is expected to spend. $Q$ is re-defined to be $Q = p \cdot Q_h + (1 - p) \cdot Q_u$. It can be easily shown that all the previous results hold: Theorems 2, 3, and 4 continue to hold with the re-defined $Q$.

5 Discussion

To the best of our knowledge, we are the first to present a model that incorporates customer satisfaction into customer value analysis. By doing so, we integrate the behavioral customer satisfaction and quantitative customer value analysis literature. This is a significant contribution because customer satisfaction is an important, if not the most important, contributor of life-time value. Also, customer life-time value is inherently tied to repeat purchases and it seems odd to ignore customer satisfaction in estimating it.

We develop our model by building on the seminal work of Schmittlein et al. (1987) and Schmittlein and Paterson (1994). This generalized model allows the purchase rate to vary with service outcome so that better service leads to a higher purchase rate. Like our previous work, we explicitly capture heterogeneity by allowing customers to have different purchase and departure rates. Consequently, the purchase rate changes both across time and across customer population in our model. (Prior research only allows purchase rate to change across population but not across time).

We derive a formula for determining the total number of expected purchases from a customer base over a time period. This formula reveals a surprising and powerful result: life-time

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4We also extend the analysis to the case where arrival and death rates are correlated. The qualitative nature of the results remains unchanged. Details are available upon request from the authors.
value has an increasing return to scale in customer satisfaction. This may explain why there is a relentless pursuit for customer satisfaction by many firms. Our formula also suggests a potential downward bias in predicting total number of purchases if one approximates the mixture Poisson processes by a single average Poisson process. This bias peaks at a mediocre customer satisfaction of 50%. Finally, we examine how the firm should optimally invest in customer satisfaction when the latter can only be achieved via costly productive processes. Since both the revenue and cost functions are convex in customer satisfaction, we determine the sufficient conditions needed for the existence of an interior optimal solution (i.e., optimal customer satisfaction is between 0 and 1).

To improve the applicability of our results, we extend our model to include discounting and a contingent spending amount. While these extensions make the formula for the total number of expected purchases more complex, they do not change the qualitative predictions of the formula. For example, the total number of expected purchases continues to be convex in customer satisfaction. Also, ignoring customer satisfaction in estimating total expected purchases can lead to a systematic downward bias.

Our model has several managerial implications. First, our model implies that it is crucial to include customer satisfaction into the prediction of total expected purchases from a customer base. This finding suggests a natural extension of the classical RFM (Recency, Frequency and Monetary value) model to the RFMS (Recency, Frequency, Monetary value, and Satisfaction) model of predicting total purchases. Second, our model yields a formula for quantifying the benefits of customer satisfaction. Firms can now use our formula to weigh the potential benefits against the costs of increasing customer satisfaction. Third, we believe our model can serve as a useful back-end engine for a CRM system since every service encounter outcome can be captured and used to modify the expected life-time value of a customer. In this way, a customer’s lifetime value can be updated dynamically and continuously to provide an accurate estimate of the value of a customer base.

\footnote{SMC’s model does not consider discounting. So this extension also increases the applicability of their model in cases where discounting is important.}
Our model opens up several new research opportunities:

1. Extending our model to include more than two levels of satisfaction will be more expansive. For example, customer satisfaction may be classified into three categories: below expectation, meets expectation, and exceeds expectation. It is also useful to empirically estimate how purchase arrival rates differ across the satisfaction categories.

2. It will be worthwhile to explore situations where other service encounters in addition to the most recent encounter influence arrival rate. For example, arrival rate may depend on a cumulative number of “happy” events. Similarly, customers may exhibit a recency bias and discount distant visits more heavily than recent visits.

3. The firm can offer distinct service classes (e.g., premium versus regular customers) based on life-time value so that the premium customer is likely to receive better service than a regular customer. Our model allows us to quantify the benefit of offering such customized service and it will be interesting to explore this explicitly in future research.

4. Currently, we assume service quality is measured by snapshot surveys of customer satisfaction in the population (i.e., cross-sectional study). Firms can potentially invest in information technology (e.g., CRM) to collect customer satisfaction data in every service encounter. If firms are allowed to offer distinct service classes, it will be interesting to assess the benefit of the additional information.

References


A Expected Number of Purchases

In this section, we prove Theorem 1. The main technique we use to study the embedded Markov Chain of the Continuous Time Markov Chain (CTMC) is the so-called “uniformization” (see Ross 1996, pp. 282-284).

Because $\lambda_2 > \lambda_1$, the uniformized rate is $\lambda_2$. Once uniformized, the CTMC spends an exponential($\lambda_2$) amount of time in each state. Moreover, the transition probabilities in the uniformized chain are as follows (state 1 for unhappy and state 2 for happy):

$$P = (P_{i,j}) = \begin{pmatrix} (1 - p) \left( \frac{\lambda_1}{\lambda_2} \right) + (1 - \frac{\lambda_1}{\lambda_2}) & 1 - p \left( \frac{\lambda_1}{\lambda_2} \right) \\ 1 - p & p \end{pmatrix}.$$  

Note that of all the transitions from state 1 into state 1, only $(1 - p)(\lambda_1/\lambda_2)$ fraction are real transitions corresponding to customer purchases; the other $1 - (1 - p)(\lambda_1/\lambda_2)$ fraction are fictitious transitions due to uniformization.

Suppose the customer starts in state $i$ and has had $n$ arrivals in the uniformized Markov chain. Because each real transition corresponds to a purchase by the customer, we would like to know how many of the $n$ arrivals correspond to real transitions. Let $N_{ij}(n)$ be the random number of real transitions into state $j$ in the first $n$ transitions of the uniformized embedded Markov chain, where $i, j \in \{1, 2\}$. Moreover, let $\tilde{N}_{ij}(n)$ be its expected value.

As an example, let’s examine $N_{11}(n)$. If the customer starts in state 1, then with probability $(1 - p)(\lambda_1/\lambda_2)$, she makes a real transition into state 1, in which case $N_{11}(n) = 1 + N_{11}(n - 1)$; with probability $(1 - \lambda_1/\lambda_2)$, she makes a dummy/fictitious transition into state 1, in which case $N_{11}(n) = 0 + N_{11}(n - 1)$; and with probability $p\lambda_1/\lambda_2$, she makes a real transition into state 2, in that case $N_{11}(n) = 0 + N_{12}(n - 1)$. So overall we have $\tilde{N}_{11}(n) = (1 - p)(\lambda_1/\lambda_2)(1 + \tilde{N}_{11}(n - 1)) + (1 - \lambda_1/\lambda_2)\tilde{N}_{11}(n - 1) + p(\lambda_1/\lambda_2)\tilde{N}_{12}(n - 1) = (1 - p)(\lambda_1/\lambda_2) + (1 - p\lambda_1/\lambda_2)\tilde{N}_{11}(n - 1) + p(\lambda_1/\lambda_2)\tilde{N}_{12}(n - 1)$. Repeating this analysis, we arrive at the following equation,

$$\begin{pmatrix} \tilde{N}_{11}(n), & \tilde{N}_{12}(n) \\ \tilde{N}_{12}(n), & \tilde{N}_{22}(n) \end{pmatrix} = \begin{pmatrix} (1 - p) \frac{\lambda_1}{\lambda_2}, & \frac{p\lambda_1}{\lambda_2} \\ 1 - p, & p \end{pmatrix} \begin{pmatrix} \tilde{N}_{11}(n), & \tilde{N}_{12}(n) \\ \tilde{N}_{12}(n), & \tilde{N}_{22}(n) \end{pmatrix}.$$  

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If we use matrix-vector notation \( \tilde{\mathbf{N}}(n) \) to denote \( \begin{pmatrix} \tilde{N}_1^1(n), & \tilde{N}_2^1(n) \\ \tilde{N}_1^2(n), & \tilde{N}_2^2(n) \end{pmatrix} \), then the above equation becomes:

\[
\tilde{\mathbf{N}}(n) = P\mathbf{W} + P\tilde{\mathbf{N}}(n-1),
\]

where \( \mathbf{W} = \begin{pmatrix} 0, & 0 \\ (1-p)/p, & 1 \end{pmatrix} \).

Noting that \( \tilde{\mathbf{N}}(0) = 0 \), or \( \tilde{\mathbf{N}}(1) = P\mathbf{W} \), we conclude:

\[
\tilde{\mathbf{N}}(n) = \left( \sum_{k=1}^{n} P^k \right) \mathbf{W}. \tag{A.2}
\]

We need to diagonalize \( P \) in order to calculate \( P^k \). First, we calculate its eigenvalues:

\[
0 = |\alpha I - P| = \begin{vmatrix} 
\alpha - 1 + \frac{p\lambda_1}{\lambda_2}, & \frac{p\lambda_1}{\lambda_2} \\
-(1-p), & \alpha - p 
\end{vmatrix} = (\alpha - 1) \left( \alpha - p \left[ 1 - \frac{\lambda_1}{\lambda_2} \right] \right) \cdot
\]

The two eigenvalues are \( \alpha_1 = 1 \) and \( \alpha_2 = p \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \). It is straightforward to calculate their corresponding eigenvectors: for \( \alpha_1 = 1 \), \( X_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \); for \( \alpha_2 = p \left( 1 - \frac{\lambda_1}{\lambda_2} \right) \), \( X_2 = \begin{pmatrix} \frac{p\lambda_1}{\lambda_2} \\ p - 1 \end{pmatrix} \).

Note that we have \( P = (X_1, X_2) \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_1 \end{pmatrix} (X_1, X_2)^{-1} \). Thus, it is easy to see that \( P^k = (X_1, X_2) \begin{pmatrix} \alpha_1^k & 0 \\ 0 & \alpha_2^k \end{pmatrix} (X_1, X_2)^{-1} \). Note that \( (X_1, X_2) = \begin{pmatrix} 1, \frac{p\lambda_1}{\lambda_2} \\ 1, \ p - 1 \end{pmatrix} \) and \( (X_1, X_2)^{-1} = \begin{pmatrix} p - 1, \ -\frac{p\lambda_1}{\lambda_2} \\ -1, \ 1 \end{pmatrix} / \Delta \), where \( \Delta = p - 1 - \frac{p\lambda_1}{\lambda_2} \). It is clear that \( 1 + \Delta = \alpha_2 \) and \( 1 - \alpha_2 = -\Delta \).

Therefore,

\[
\tilde{\mathbf{N}}(n) = \left( \sum_{k=1}^{n} P^k \right) \mathbf{W}
= (X_1, X_2) \begin{pmatrix} \sum_{k=1}^{n} \alpha_1^k \\ \sum_{k=1}^{n} \alpha_2^k \end{pmatrix} (X_1, X_2)^{-1} \mathbf{W},
\]

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We first condition on \( n \), the number of transitions in the uniformized Markov chain. If a customer starts in state 1, then the expected number of purchases (among the \( n \) transitions) is:

\[
\mathbb{N}_1^1(n) + \mathbb{N}_2^2(n) = \left[ \frac{-n(1-p)\lambda_1}{\lambda_2} + \frac{(1-p)\lambda_1}{\lambda_2} \left( \frac{a_2 - a_2^{n+1}}{1 - a_2} \right) + \frac{-np\lambda_1}{\lambda_2} + \frac{p\lambda_1}{\lambda_2} \left( \frac{a_2 - a_2^{n+1}}{1 - a_2} \right) \right] / \Delta = -K_1 n + K_2 a_2 - K_2 a_2^{n+1},
\]

where \( K_1 = \frac{\lambda_1}{\lambda_2} \Delta = \frac{-\lambda_1}{(1-p)\lambda_2 + p\lambda_1} = -\frac{\lambda_1}{\lambda} \) and \( K_2 = \frac{\lambda_1}{\lambda_2 \Delta (1 - a_2)} = \frac{-\lambda_1 \lambda_2}{[(1-p)\lambda_2 + p\lambda_1]^2} = -\frac{\lambda_1 \lambda_2}{\lambda^3} \).

Next, we calculate the unconditioned expected number of purchases:

\[
\int_0^T \sum_{t=0}^{\infty} \left[ (-K_1 n + K_2 a_2 - K_2 a_2^{n+1}) e^{-\lambda_2 t (\lambda_2 t)^n} \right] n! \mu e^{-\mu t} dt
\]

\[
+ \int_T^\infty \sum_{t=0}^{\infty} \left[ (-K_1 n + K_2 a_2 - K_2 a_2^{n+1}) e^{-\lambda_2 t (\lambda_2 T)^n} \right] n! \mu e^{-\mu t} dt
\]

\[
= \int_0^T \left[ -K_1 \lambda_2 t + K_2 a_2 - K_2 a_2 e^{-(1-a_2)\lambda_2 t} \right] \mu e^{-\mu t} dt
\]

\[
+ \int_T^\infty \left[ -K_1 \lambda_2 t + K_2 a_2 - K_2 a_2 e^{-(1-a_2)\lambda_2 T} \right] \mu e^{-\mu t} dt
\]

\[
= -K_1 \lambda_2 \left[ 1 - e^{-\mu T} (1 + \mu T) \right] + K_2 a_2 \left( 1 - e^{-\mu T} - K_2 a_2 \mu \right) \frac{-1 + e^{T(-\lambda_2 + a_2 \lambda_2 - \mu)}}{-\lambda_2 + a_2 \lambda_2 - \mu}
\]

\[
+ \left[ -K_1 \lambda_2 T + K_2 a_2 - K_2 a_2 e^{-(1-a_2)\lambda_2 T} \right] e^{-\mu T}
\]

\[
= -K_1 \lambda_2 \left[ 1 - e^{-\mu T} \right] + K_1 \lambda_2 T e^{-\mu T} + K_2 a_2 \left( 1 - e^{-\mu T} - K_2 a_2 \mu \frac{1 - e^{-(\lambda + \mu)T}}{\lambda + \mu} \right) - K_1 \lambda_2 T e^{-\mu T} + K_2 a_2 e^{-\mu T} - K_2 a_2 e^{-(\lambda + \mu)T},
\]

where \( \lambda = (1-p)\lambda_2 + p\lambda_1 = (1-a_2)\lambda_2 \).

By definitions of \( K_1 \), \( K_2 \), and \( a_2 \), we have \( K_1 \lambda_2 = -\frac{\lambda_1 \lambda_2}{\lambda^2} \) and \( K_2 a_2 = -\frac{\mu \lambda_1 (\lambda_2 - \lambda_1)}{\lambda^2} \). Therefore,
for, (A.5) can be simplified to:

\[
\frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T}) + \frac{p\lambda_1 (\lambda_1 - \lambda_2)}{\lambda (\lambda + \mu)} \left(1 - e^{-(\lambda + \mu)T}\right).
\]

Because the purchase amount is independent of the number of purchases, the total expected purchase amount is a simple product of the two averages:

\[
\pi_u = Q \left[ \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu T}) - \frac{p\lambda_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu)T}) \right].
\]

A.2 Customer Starts “Happy”

Similarly, we first condition on \(n\), the number of transitions in the uniformized Markov chain. If a customer starts in state 1, then the expected number of purchase visits (among the \(n\) transitions) is:

\[
\bar{N}_1^2(n) + \bar{N}_2^2(n) = \left[ \frac{-n(1-p)\lambda_1}{\lambda_2} - \frac{(1-p)^2}{p} \left( \alpha_2 - \frac{\alpha_2^{n+1}}{1 - \alpha_2} \right) \right] / \Delta
\]

\[
= \left[ \frac{-n\lambda_1}{\lambda_2} + \frac{p-1}{p} \left( \alpha_2 - \frac{\alpha_2^{n+1}}{1 - \alpha_2} \right) \right] / \Delta
\]

\[
= -K_1 n + K_3 \alpha_2 - K_3 \alpha_2^{n+1},
\]

(A.6)

where \(K_3 = \frac{p-1}{p\Delta(1-\alpha_2)} = \frac{(1-p)\lambda_2^2}{p\lambda_2^2}\).

Next, we calculate the unconditioned expected number of purchase during \([0, T]\):

\[
\int_{t=0}^{T} \sum_{n=0}^{\infty} \left[ (-K_1 n + K_3 \alpha_2 - K_3 \alpha_2^{n+1}) \frac{e^{-\lambda_2 t} (\lambda_2 t)^n}{n!} \right] \mu e^{-\mu t} \, dt
\]

\[
+ \int_{t=T}^{\infty} \sum_{n=0}^{\infty} \left[ (-K_1 n + K_3 \alpha_2 - K_3 \alpha_2^{n+1}) \frac{e^{-\lambda_2 T} (\lambda_2 T)^n}{n!} \right] \mu e^{-\mu t} \, dt
\]

\[
= \int_{t=0}^{T} \left[ -K_1 \lambda_2 t + K_3 \alpha_2 - K_3 \alpha_2 e^{-(1-\alpha_2)\lambda_2 t} \right] \mu e^{-\mu t} \, dt
\]

\[
+ \int_{t=T}^{\infty} \left[ -K_1 \lambda_2 T + K_3 \alpha_2 - K_3 \alpha_2 e^{-(1-\alpha_2)\lambda_2 T} \right] \mu e^{-\mu t} \, dt
\]

\[
= -K_1 \lambda_2 \frac{1 - e^{-\mu T} (1 + \mu T)}{\mu} + K_3 \alpha_2 (1 - e^{-\mu T}) - K_3 \alpha_2 \mu \left[ -1 + e^{(\lambda_2 + \alpha_2 \lambda_2 - \mu)} \right] - \lambda_2 + \alpha_2 \lambda_2 - \mu
\]

\[
+ \left[ -K_1 \lambda_2 T + K_3 \alpha_2 - K_3 \alpha_2 e^{-(1-\alpha_2)\lambda_2 T} \right] e^{-\mu T}
\]

\[
= -K_1 \lambda_2 \frac{1 - e^{-\mu T}}{\mu} + K_1 \lambda_2 e^{-\mu T} + K_3 \alpha_2 (1 - e^{-\mu T}) - K_3 \alpha_2 \frac{\mu}{\lambda + \mu} \left(1 - e^{-(\lambda + \mu)T}\right)
\]

\[
- K_1 \lambda_2 e^{-\mu T} + K_3 \alpha_2 e^{-\mu T} - K_3 \alpha_2 e^{-(\lambda + \mu)T}
\]

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The part regarding heterogeneity across the population is given by:

\[ \pi_u = Q \int_0^\infty \int_0^\infty \left[ V_1 - V_2 \right] \times \left[ g_{\lambda_2}(\lambda_2; r, \alpha) \times g_{\mu}(\mu; s, \beta) \right] d\lambda_2 d\mu. \]

This concludes the proof of Theorem 1.

## B Customer Heterogeneity

### B.1 Customer Starts “Unhappy”

In this section, we prove Theorems 2 and 5. (4.1) can be rewritten as follows:

\[ \pi_u = Q \left[ V_1 - V_2 \right], \quad \text{(B.1)} \]

where

\[ V_1 = \sum_{i=0}^{K-1} d^i e^{-\kappa r T_0} \frac{k \lambda_2}{p_k \mu} \left( 1 - e^{-\mu T_0} \right) \]

and

\[ V_2 = \sum_{i=0}^{K-1} d^i e^{-\kappa r (p_k \lambda_2 + \mu)} \frac{p_k (1-k)}{p_k \lambda_2 + \mu} \left( 1 - e^{-(p_k \lambda_2 + \mu) T_0} \right). \]

The expression of the (discounted) expected number of real purchases which incorporates heterogeneity across the population is given by:

\[ \pi_h = Q \left[ V_1 - V_2 \right] \left[ g_{\lambda_2}(\lambda_2; r, \alpha) \times g_{\mu}(\mu; s, \beta) \right] d\lambda_2 d\mu. \]

The part regarding \( V_1 \) in (B.2) is given by:

\[ \int_0^\infty \int_0^\infty \left[ V_1 \right] \times \left[ g_{\lambda_2}(\lambda_2; r, \alpha) \times g_{\mu}(\mu; s, \beta) \right] d\lambda_2 d\mu = \frac{k}{p_k} \sum_{i=0}^{K-1} d^i \int_0^\infty \frac{\lambda_2^r e^{-\alpha \lambda_2}}{\Gamma(r)} \int_0^\infty \frac{\mu^{s-2} \beta^s e^{-(\beta + i) T_0} \mu}{\Gamma(s)} \frac{\mu^{s-2} \beta^s e^{-(\beta + (i+1) T_0) \mu}}{\Gamma(s)} d\mu d\lambda_2 \]

\[ = \frac{k}{p_k} \frac{r \beta}{\alpha (s-1)} \sum_{i=0}^{K-1} d^i \left[ \left( \frac{\beta}{\beta + i T_0} \right)^{s-1} - \left( \frac{\beta}{\beta + (i+1) T_0} \right)^{s-1} \right]. \]
To evaluate the part regarding $V_2$ in (B.2), it is convenient to consider separately the two cases: $p_k \neq \frac{a}{\beta}$ and $p_k = \frac{a}{\beta}$.

**Case 1: $p_k \neq \frac{a}{\beta}$**. Let us change variables: $z = p_k \lambda_2 + \mu$ and $q = \frac{p_k \lambda_2}{p_k \lambda_2 + \mu}$. The Jacobian is $\frac{1}{p_k}$. Then

$$\int_0^\infty \int_0^\infty \left[ V_2 \right] \times \left[ g_{\lambda_2}(\lambda_2; r, \alpha) \times g_\mu(\mu; s, \beta) \right] d\lambda_2 d\mu$$

$$= \frac{p_k(1-k)}{p_k} \frac{\alpha^\beta \Gamma(r) \Gamma(s)}{\Gamma(r+s)} \sum_{i=0}^{K-1} \frac{d^i}{(\beta+iT_0)^{r+s}} \int_0^\infty \int_0^\infty \left\{ \frac{\lambda_2^i \mu^{s-1} e^{-(\alpha+p_k iT_0)\lambda_2 -(\beta+iT_0)\mu}}{(p_k \lambda_2 + \mu)^{s} \lambda_2^{(s-1)}} \left( p_k \lambda_2 + \mu \right)^{-(s-1)} \right\} d\lambda_2 d\mu$$

$$= \frac{p_k(1-k)}{p_k} \frac{\alpha^\beta \Gamma(r+s)}{\Gamma(r) \Gamma(s)} \sum_{i=0}^{K-1} \frac{d^i}{(\beta+iT_0)^{r+s}} \int_0^\infty \left[ (1-q)^{s-1} \left( 1 - \frac{p_k \beta - \alpha}{p_k (\beta+iT_0)} \right) q^{-(r+s)} dq \right] \left[ (1-q)^{s-1} \left( 1 - \frac{p_k \beta - \alpha}{p_k (\beta+iT_0)} \right) q^{-(r+s)} dq \right]$$

$$= \frac{p_k(1-k)}{p_k} \frac{\alpha^\beta \Gamma(r+s)}{\Gamma(r) \Gamma(s)} \sum_{i=0}^{K-1} \frac{d^i}{(\beta+iT_0)^{r+s}} F \left( r+s, r+1; r+s+1; \frac{p_k \beta - \alpha}{p_k (\beta+iT_0)} \right)$$

$$= \frac{p_k(1-k)}{p_k} \frac{\alpha^\beta \Gamma(r+s)}{\Gamma(r) \Gamma(s)} \sum_{i=0}^{K-1} \frac{d^i}{(\beta+iT_0)^{r+s}} F \left( r+s, r+1; r+s+1; \frac{p_k \beta - \alpha}{p_k (\beta+iT_0)} \right)$$

Note that the integral in (B.4) is written as a Gauss hypergeometric function (Abramowitz and Stegun 1972, p. 558). We get the following by putting (B.3) and (B.4) together:

$$\pi_u = Q \times \left\{ \frac{kr \beta}{p_k \alpha (s-1)} \sum_{i=0}^{K-1} d^i \left[ \frac{(\beta+iT_0)^{s-1}}{(\beta+(i+1)T_0)^{s-1}} - \frac{(\beta+(i+1)T_0)^{s-1}}{F(a, b; c; z_1(i))} \right] \right\}$$

where $a = r + s$, $b = r + 1$, $c = r + s + 1$, $z_1(i) = \frac{p_k \beta - \alpha}{p_k (\beta+iT_0)}$, and $z_2(i) = \frac{p_k \beta - \alpha}{p_k (\beta+(i+1)T_0)}$. If we let $d = 1$, (B.5) becomes:

$$\pi_u = Q \times \left\{ \frac{kr \beta}{p_k \alpha (s-1)} \sum_{i=0}^{K-1} d^i \left[ \frac{(\beta+iT_0)^{s-1}}{(\beta+(i+1)T_0)^{s-1}} - \frac{(\beta+(i+1)T_0)^{s-1}}{F(a, b; c; z_1)} \right] \right\}$$

where $a = r + s$, $b = r + 1$, $c = r + s + 1$, $z_1 = \frac{p_k \beta - \alpha}{p_k \beta}$, and $z_2 = \frac{p_k \beta - \alpha}{p_k (\beta+T)}$.

**Case 2: $p_k = \frac{a}{\beta}$**. In this case, the expression is greatly simplified.

$$\int_0^\infty \int_0^\infty \left[ V_2 \right] \times \left[ g_{\lambda_2}(\lambda_2; r, \alpha) \times g_\mu(\mu; s, \beta) \right] d\lambda_2 d\mu$$
Putting (B.3) and (B.7) together, we get the following:

\[
\begin{align*}
\pi_u &= Q \times \left\{ - \frac{kr \beta}{pk \alpha} \sum_{i=0}^{K-1} d^i \left[ \frac{\beta}{\beta+iT_0} \right]^{s-1} - \frac{1}{(\beta+iT_0)^{r+s}} + \frac{1}{(\beta+i+1)T_0)^{r+s}} \right\}, \\
\pi_u &= Q \times \left\{ - \frac{kr \beta}{pk \alpha} \sum_{i=0}^{K-1} d^i \left[ \frac{1}{(\beta+iT_0)^{r+s}} - \frac{1}{(\beta+i+1)T_0)^{r+s}} \right] \right\}. 
\end{align*}
\]

If we let \( d = 1 \), (B.8) becomes:

\[
\pi_u = Q \times \left\{ - \frac{kr \beta}{pk \alpha} \left[ 1 - \left( \frac{1}{\beta+T} \right)^{s-1} \right] - \frac{1}{\beta^{r+s}} + \frac{1}{(\beta+1)^{r+s}} \right\}. 
\]

### B.2 Customer Starts “Happy”

The derivation of the (discounted) expected number of real purchases for customers starting happy is virtually identical to that of customers starting unhappy. The only difference in these two expressions is \( pk(1-k) \) for dissatisfied customers and \((1-p)(1-k) \) for satisfied customers after the plus sign.

This concludes the proof of Theorems 2 and 5.

### C Revenue, Cost, and Profit Function

#### C.1 Revenue Function

We note that

\[
\pi_h = \pi_u + Q \lambda_2 - \lambda_1 \frac{\lambda_2 - \lambda_1}{\lambda + \mu} \left( 1 - e^{-\lambda + \mu T} \right). 
\]
Lemma 1

(i) The first three derivatives of \( \frac{\lambda_2 - \lambda_1}{\lambda + \mu} (1 - e^{-(\lambda + \mu)T}) \) are non-negative.

(ii) The first three derivatives of \( \pi_u \) and \( \pi_h \) are non-negative.

Proof

Part (i) Let \( f(x) = \frac{1 - e^{-Tx}}{x} \), we have

- \( f'(x) = \frac{Te^{-Tx}x - (1 - e^{-Tx})}{x^2} \). If we let \( g(x) = Te^{-Tx}x - (1 - e^{-Tx}) \), then \( g(0) = 0 \) and \( g'(x) = -T^2xe^{-Tx} \leq 0 \). So \( g(x) \leq 0, \) and \( f'(x) \leq 0, \) for \( x \geq 0. \) Therefore, \( f(\lambda + \mu) \) has a positive first derivative with respect to \( p, \) because \( \lambda'(p) = -(\lambda_2 - \lambda_1) \leq 0. \)

- \( f''(x) = \frac{2e^{-Tx}(T^2x^2 + 2Tx + 2)}{x^3}. \) If we let \( g(x) = 2 - e^{-Tx}(T^2x^2 + 2Tx + 2) \), then \( g(0) = 0. \) Moreover, \( g'(x) = T^3xe^{-Tx} \geq 0. \) So \( g(x) \geq 0, \) and \( f''(x) \geq 0, \) for \( x \geq 0. \) It follows that \( f''(\lambda + \mu) \) has a positive second derivative with respect to \( p. \)

- \( f'''(x) = \frac{-6 + e^{-Tx}[T^3x^3 + 3T^2x^2 + 6T + 6]}{x^4}. \) If we let \( g(x) = -6 + e^{-Tx} [T^3x^3 + 3T^2x^2 + 6T + 6], \) then \( g(0) = 0 \) and \( g'(x) = -T^4x^3e^{-Tx} \leq 0 \) for \( x \geq 0. \) So \( g(x) \leq 0, \) and \( f'''(x) \leq 0, \) for \( x \geq 0. \) It follows that \( f'''(\lambda + \mu) \) has a positive third derivative with respect to \( p. \), because \( \lambda'(p) = -(\lambda_2 - \lambda_1) \) is a non-positive constant.

Part (ii) From Appendix A, we know that

\[
\pi_u = Q \int_0^T \sum_{n=0}^\infty \left[ \left( \bar{N}_1(n) + \bar{N}_2(n) \right) \frac{e^{-\lambda_2 t} (\lambda_2 t)^n}{n!} \right] \mu e^{-\mu t} \, dt
+ Q \int_0^\infty \sum_{n=0}^\infty \left[ \left( \bar{N}_1(n) + \bar{N}_2(n) \right) \frac{e^{-\lambda_2 T} (\lambda_2 T)^n}{n!} \right] \mu e^{-\mu t} \, dt.
\]

Therefore, it suffices to show that the first three derivatives of \( \bar{N}_1(n) + \bar{N}_2(n) \) are convex in \( p \) for all \( n. \) Also from Appendix A:

\[
\bar{N}_1(n) + \bar{N}_2(n) = \frac{\lambda_1}{\lambda_2 \Delta} \left( \sum_{k=1}^n \alpha_2^k - n \right) = \frac{\lambda_1}{\lambda_2} \sum_{k=1}^n \left( \frac{\alpha_2^k - 1}{\alpha_2 - 1} \right) = \frac{\lambda_1}{\lambda_2} \sum_{k=1}^n \left( \sum_{i=0}^{k-1} \alpha_2^i \right).
\]
Because any derivative of \( \alpha_i^j = p^j \left( 1 - \frac{\lambda_i}{\lambda} \right)^i \) is non-negative with respect to \( p \) as long as \( i \geq 0 \), the sum clearly has non-negative first three derivatives with respect to \( p \).

Because \( \pi_h = \pi_u + Q \frac{\lambda_2 - \lambda_1}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)T} \right) \), it follows easily that \( \pi_h \) also has non-negative first three derivatives with respect to \( p \).

\[
\pi'(p) = p\pi'_h(p) + (1-p)\pi'_u(p) + (\pi_h(p) - \pi_u(p))
\]

\[
= p\pi'_h(p) + (1-p)\pi'_u(p) + Q \left( \frac{\lambda_2 - \lambda_1}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)T} \right) \right) \geq 0
\]

\[
\pi''(p) = p\pi''_h(p) + (1-p)\pi''_u(p) + 2(\pi_h(p) - \pi_u(p))'
\]

\[
= p\pi''_h(p) + (1-p)\pi''_u(p) + 2Q \left( \frac{\lambda_2 - \lambda_1}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)T} \right) \right)' \geq 0
\]

\[
\pi'''(p) = p\pi'''_h(p) + (1-p)\pi'''_u(p) + 3(\pi_h(p) - \pi_u(p))''
\]

\[
= p\pi'''_h(p) + (1-p)\pi'''_u(p) + 3Q \left( \frac{\lambda_2 - \lambda_1}{\lambda + \mu} \left( 1 - e^{-(\lambda + \mu)T} \right) \right)'' \geq 0
\]

because each term is non-negative by parts(i) and (ii).

\[\square\]

### C.2 NPV Revenue Function

\[
\pi = p\pi_2 + (1-p)\pi_1 = QH \frac{\lambda_1}{\lambda \mu} \left( 1 - e^{-\mu T_0} \right) + QG \frac{p(1-p)(\lambda_2 - \lambda_1)^2}{\lambda (\lambda + \mu)} \left( 1 - e^{-(\lambda + \mu)T_0} \right)
\]

\[
= GA + Q(H - G) \frac{\lambda_1}{\lambda \mu} \left( 1 - e^{-\mu T_0} \right), \quad \text{(C.2)}
\]

where

\[
A = Q \left[ \frac{\lambda_1}{\lambda \mu} \left( 1 - e^{-\mu T_0} \right) + \frac{p(1-p)(\lambda_2 - \lambda_1)^2}{\lambda (\lambda + \mu)} \left( 1 - e^{-(\lambda + \mu)T_0} \right) \right].
\]

is exactly the the undiscounted revenue function (2.3) with \( T \) replaced by \( T_0 \). We also know that \( A' \geq 0, A'' \geq 0 \) and \( A''' \geq 0 \) (all derivatives with respect to \( p \)).

Because \( G = \sum_{i=0}^{K-1} (e^{-(\lambda + \mu)T_0} d)^i \), we have \( \frac{dG}{d\lambda} \leq 0, \frac{d^2G}{d\lambda^2} \geq 0, \) and \( \frac{d^3G}{d\lambda^3} \leq 0 \). Furthermore, because \( \lambda = \lambda_2 - (\lambda_2 - \lambda_1)p \) and \( \frac{d\lambda}{dp} \leq 0 \), we have \( \frac{dG}{dp} \geq 0, \frac{d^2G}{dp^2} \geq 0, \) and \( \frac{d^3G}{dp^3} \geq 0 \). Therefore,

\[
(GA)' = G'A + GA' \geq 0;
\]

\[
(GA)'' = G''A + 2G'A' + GA'' \geq 0;
\]

\[
(GA)''' = G'''A + 3G''A' + 3G'A'' + GA''' \geq 0. \quad \text{(C.3)}
\]
Furthermore,
\[
(H - G) \frac{\lambda_1 \lambda_2}{\lambda^2} (1 - e^{-\mu T_0}) = \sum_{i=0}^{K-1} (e^{-\mu T_0} d^i) (1 - e^{-i \lambda T_0}) \frac{\lambda_1 \lambda_2}{\lambda^2} (1 - e^{-\mu T_0})
\]
\[
= \sum_{i=0}^{K-1} \left(M_i \cdot \frac{1 - e^{-i \lambda T_0}}{\lambda}\right),
\]
where \(M_i\)'s are constants.

To show that \(\pi\) has non-negative first three derivatives with respect to \(p\), by (C.2) it suffices to show that the first three derivatives of (C.4) with respect to \(p\) are non-negative. Because we have \(\lambda'(p) \leq 0\), it suffices to show that the signs of the first three derivatives of (C.4) with respect to \(\lambda\) are alternating. The following lemma provides this result.

\[\text{Lemma 2} \quad \text{Let } f(x) = \frac{1 - e^{-ax}}{x}, \text{ where } a \geq 0. \text{ Then, for } x \geq 0, f'(x) \leq 0, f''(x) \geq 0, \text{ and } f'''(x) \leq 0.\]

\[\text{Proof}\]
\[
f'(x) = \frac{e^{-ax}}{x^2} [(ax + 1) - e^{ax}] = \frac{e^{-ax}}{x^2} f_1(x),
\]
where \(f_1(x) = (ax + 1) - e^{ax}\). Because \(f_1(0) = 0\) and \(f'_1(x) = a(1 - e^{ax}) \leq 0, \forall a, x \geq 0\), we must have \(f_1(x) \leq 0\), and hence \(f'(x) \leq 0\), for all \(x \geq 0\).

\[
f''(x) = \frac{e^{-ax}}{x^3} \left[2e^{ax} - (2 + 2ax + a^2 x^2)\right] = \frac{e^{-ax}}{x^3} f_2(x),
\]
where \(f_2(x) = 2e^{ax} - (2 + 2ax + a^2 x^2)\). Because \(f_2(0) = 0\) and \(f'_2(x) = 2a(e^{ax} - (1 + ax)) = -2af_1(x) \geq 0 \forall a, x \geq 0\), we must have \(f_2(x) \geq 0\), and hence \(f''(x) \geq 0\), for all \(x \geq 0\).

\[
f'''(x) = \frac{e^{-ax}}{x^4} \left[6 + 6ax + 3a^2 x^2 + a^3 x^3 - 6e^{ax}\right] = \frac{e^{-ax}}{x^4} f_3(x),
\]
where \(f_3(x) = 6 + 6ax + 3a^2 x^2 + a^3 x^3 - 6e^{ax}\). Because \(f_3(0) = 0\) and \(f'_3(x) = 3a(2 + 2ax + a^2 x^2 - 2e^{ax}) = -3af_2(x) \leq 0 \forall a, x \geq 0\), we must have \(f_3(x) \leq 0\), and hence \(f'''(x) \leq 0\), for all \(x \geq 0\).

\[\square\]

\section*{C.3 Profit Function}

In this section, we prove Theorem 3.

\[\text{Proof}\]

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Part (i) If we let \( B_3 = \pi''(1) \), then because \( \pi''(p) \geq 0 \), we have \( \pi''(p) \leq B_3 \). Moreover, that \( \pi'' \) is bounded implies that \( \pi' \) is finite on \( (0, 1] \). Therefore, if we let \( B_1 = \pi'(0) \) and \( B_2 = \pi'(1) \), then we have \( B_1 \leq \pi'(p) \leq B_2, \forall p \in (0, 1] \), because \( \pi'(p) \) is non-decreasing.

Because the probability distribution functions of \( \lambda_1, \lambda_2, \) and \( \mu \) are exponentially decaying when any of the variables go to infinity, and \( B_1 \) and \( B_2 \) do not increase faster than linearly, the integrals converge at \( \infty \).

Part (ii) We check the three conditions one by one. First,

\[
\pi''(p) = NQ \int \pi''(p) \, d\Sigma(\lambda_1, \lambda_2, \mu) - C''(p) \\
\leq NQ \int B_3 \, d\Sigma(\lambda_1, \lambda_2, \mu) - C''(p)|_{p=0} = NQ \int B_3 \, d\Sigma(\lambda_1, \lambda_2, \mu) - \omega(\omega + 1)M < 0,
\]

where the inequality holds due to part [(i)] and the fact that \( C''(p) \) is increasing in \( p \).

Second,

\[
\pi'(p)|_{p=0} = NQ \int \pi'(p) \, d\Sigma(\lambda_1, \lambda_2, \mu)|_{p=0} - C'(p)|_{p=0} \geq NQ \int B_1 \, d\Sigma(\lambda_1, \lambda_2, \mu) - M \omega > 0.
\]

Moreover, \( \int \pi'(p) \, d\Sigma(\lambda_1, \lambda_2, \mu) \leq \int B_2 \, d\Sigma(\lambda_1, \lambda_2, \mu) \), which is finite. As \( p \to 1 \), \( C'(p) \to \infty \).

So we must have \( \pi'_a(p) \to -\infty \).

Finally, for condition 3, it suffices to show that \( \pi_a(p)|_{p=0} > 0 \).

\[
\pi_a(p)|_{p=0} = NQ \int \pi(0) \, d\Sigma(\lambda_1, \lambda_2, \mu) - C(0) = NQ\pi(0) - M > 0.
\]

When the optimal service quality is an interior point, “marginal revenue = marginal cost” at the optimal point. The discounting of revenue, however, effectively reduces the marginal revenue at all the points (see (4.3)), while the marginal cost remains the same. Therefore, the optimal service quality level is lower with discounting than without discounting.

C.4 NPV Calculation

In this section, we prove Theorem 4. Equations (2.1) and (2.2) give us the total undiscounted expected purchases by a customer during any \( T_0 \) sub-interval:
• When the customer is unhappy at the beginning of the sub-interval:

\[ \pi_1 = Q \left[ \frac{\lambda_1 \lambda_2}{\lambda \mu} \left( 1 - e^{-\mu T_0} \right) - \frac{p \lambda_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} \left( 1 - e^{-(\lambda + \mu) T_0} \right) \right]. \quad \text{(C.5)} \]

• When the customer is happy at the beginning of the sub-interval:

\[ \pi_2 = Q \left[ \frac{\lambda_1 \lambda_2}{\lambda \mu} \left( 1 - e^{-\mu T_0} \right) + \frac{(1 - p) \lambda_2 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} \left( 1 - e^{-(\lambda + \mu) T_0} \right) \right]. \quad \text{(C.6)} \]

Let’s examine the discount from period to period. The state of the customer at the beginning of each sub-interval is a Markov chain, and its transition matrix \( V \) can be calculated as follows (assuming that the customer does not “die” in that sub-interval):

\[
V = \sum_{k=0}^{\infty} \frac{e^{-\lambda_2 T_0} (\lambda_2 T_0)^k}{k!} p^k
= \sum_{k=0}^{\infty} \frac{e^{-\lambda_2 T_0} (\lambda_2 T_0)^k}{k!} (X_1, X_2) \begin{pmatrix} 1 \\ \alpha_k \end{pmatrix} (X_1, X_2)^{-1}
= (X_1, X_2) \begin{pmatrix} 1 \\ e^{-\lambda T_0} \end{pmatrix} (X_1, X_2)^{-1}. \quad \text{(C.7)}
\]

Let \( S_i \in \{u, h\} \) be the state of the customer at the beginning of time interval \([iT_0, (i+1)T_0]\). Moreover, let \( p_{S_i} = (p_{i,u}, p_{i,h}) \), where \( p_{i,u} \) is the probability of being unhappy at the beginning of \( i \)th time interval, and \( p_{i,h} \) is the probability of being happy at the beginning of \( i \)th time interval. Since \( e^{-\mu T_0} \) is the probability that a customer “survives” a sub-interval, the total NPV of a customer for \( K \) sub-intervals can be calculated as follows:

\[
NPV = \begin{pmatrix} \pi_u \\ \pi_h \end{pmatrix} = \pi_{S_0} + e^{-\mu T_0} d(\pi_{S_1} + e^{-\mu T_0} d(\pi_{S_2} \ldots ))
\]

\[
= \sum_{i=0}^{K-1} (e^{-\mu T_0} d)^i \pi_{S_i}
\]

\[
= \sum_{i=0}^{K-1} (p_{S_0} (e^{-\mu T_0} d)^i V)^i \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}
\]

\[
= p_{S_0} \sum_{i=0}^{K-1} (X_1, X_2) \begin{pmatrix} (e^{-\mu T_0} d)^i \\ (e^{-(\lambda + \mu) T_0} d)^i \end{pmatrix} (X_1, X_2)^{-1} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix}
\]

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\[
\begin{align*}
&= p_S \mathbf{0} \begin{pmatrix} 1, \frac{pS_1}{\lambda_2} \\ 1, p-1 \end{pmatrix} \begin{pmatrix} H \\ G \end{pmatrix} \begin{pmatrix} p-1, \ -\frac{pS_1}{\lambda_2} \\ -1, \ 1 \end{pmatrix} \begin{pmatrix} \pi_1 \\ \pi_2 \end{pmatrix} / \Delta \\
&= p_S \mathbf{0} \begin{pmatrix} Q \begin{pmatrix} H \frac{(p-1)\pi_1 - \frac{pS_1}{\lambda_2}}{\Delta} + \frac{pS_1}{\lambda_2} G (\pi_1 - \pi_2) \\ Q \begin{pmatrix} \frac{(p-1)\pi_1 - \frac{pS_1}{\lambda_2}}{\Delta} + \frac{b-1}{\Delta} G (\pi_1 - \pi_2) \end{pmatrix} \end{pmatrix} \\
&= p_S \mathbf{0} \begin{pmatrix} Q \begin{pmatrix} H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu t_0}) - G \frac{pS_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu) t_0}) \\ Q \begin{pmatrix} H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu t_0}) + G \frac{(1-p) \lambda_2 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu) t_0}) \end{pmatrix} \end{pmatrix}. 
\end{align*}
\]

So,

\[
\begin{align*}
\pi_u &= Q \begin{pmatrix} H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu t_0}) - G \frac{pS_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu) t_0}) \\ Q \begin{pmatrix} H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu t_0}) + G \frac{(1-p) \lambda_2 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu) t_0}) \end{pmatrix} \end{pmatrix}, \\
\pi_h &= Q \begin{pmatrix} H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu t_0}) + G \frac{(1-p) \lambda_2 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu) t_0}) \\ Q \begin{pmatrix} H \frac{\lambda_1 \lambda_2}{\lambda \mu} (1 - e^{-\mu t_0}) - G \frac{pS_1 (\lambda_2 - \lambda_1)}{\lambda (\lambda + \mu)} (1 - e^{-(\lambda + \mu) t_0}) \end{pmatrix} \end{pmatrix}. 
\end{align*}
\]

It is straightforward to verify that \( \pi = p\pi_h + (1-p)\pi_u. \)

This concludes the proof of Theorem 4.
Figure 1: Expected Number of Purchases
Figure 2: Optimal Customer Satisfaction as a Function of $M$ ($k = 0.50, \omega = 3.0$)

Figure 3: Optimal Profit as a Function of $M$ ($k = 0.50, \omega = 3.0$)
Figure 4: Optimal Customer Satisfaction as a Function of $\omega$ ($k = 0.50, M = .001$)

Figure 5: Optimal Profit as a Function of $\omega$ ($k = 0.50, M = .001$)